

$$1. (x^2 + y^2 + xe^y) \frac{dy}{dx} + 2xy + e^y - 1 = 0$$

$$\frac{dx(x^2 + y^2 + xe^y)dy}{dx} = (1 - e^y - 2xy)dx$$

$$(x^2 + y^2 + xe^y)dy + (-1 + e^y + 2xy)dx = 0$$

$$M(x,y) = -1 + e^y + 2xy$$

$$N(x,y) = x^2 + y^2 + xe^y$$

$$\frac{\partial M}{\partial y} = +e^y + 2x$$

$$\frac{\partial N}{\partial x} = 2x + e^y$$

A function is exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = 2x + e^y$$

Its solution $\int M dx = \int (e^y + 2xy - 1) dx = xe^y + \frac{2x^2}{2}y - x + \phi(y)$

$$= xe^y + x^2y - x + \phi(y)$$

$$\frac{\partial}{\partial y} [xe^y + x^2y - x + \phi(y)] = N$$

$$xe^y + x^2 + \phi'(y) = x^2 + y^2 + xe^y$$

$$\phi'(y) = y^2$$

$$\phi(y) = \int \phi'(y) = \frac{y^3}{3}$$

Therefore solution = $xe^y + x^2y - x + \frac{y^3}{3}$

2. $\frac{dy}{dx} = \frac{2xy - y^2}{x^2}$, this function is homogeneous because it can be written in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$$\frac{dy}{dx} = \frac{2xy}{x^2} - \frac{y^2}{x^2} = 2\frac{y}{x} - \left(\frac{y}{x}\right)^2$$

$$\frac{dy}{dx} = 2\frac{y}{x} - \left(\frac{y}{x}\right)^2 \text{ therefore homogeneous}$$



TECNO
We let $y = vx$
SPARK

$$y = v + x \frac{dv}{dx}$$

$$\frac{dy}{dx} = 2\frac{y}{x} - \left(\frac{y}{x}\right)^2 \quad \Rightarrow \frac{y}{x} = v$$

$$v + x \frac{dv}{dx} = 2v - v^2$$

$$x \frac{dv}{dx} = 2v - v^2 - v$$

$$x \frac{dv}{dx} = (v - v^2) dx$$

$$\frac{x dx dv}{xv - v^2} = \frac{(v - v^2) dx}{v - v^2}$$

$$\int \frac{1}{v - v^2} dv = \int \frac{1}{x} dx$$

$$\int \frac{1}{v} dv - \int \frac{1}{v^2} dv = \int \frac{1}{x} dx$$

$$\ln v - \frac{1}{v} = \ln x + \ln c$$

$$\ln v - \frac{1}{v} = \ln x + \ln c$$

$$\ln v - (\ln x + \ln c) = \frac{1}{v}$$

$$\ln \left[\frac{v}{xc} \right] = \frac{1}{v}$$

$$\frac{v}{xc} = e^{\frac{1}{v}}$$

$$v = xc e^{\frac{1}{v}}$$

$$\frac{y}{x} = Ax e^{\frac{x}{y}} = Ax e^{\frac{x}{y}}$$

$$y = Ax^2 e^{\frac{x}{y}}$$

3. $x(0) = 100$

$S(t)$ = mass of salt in tank at time $t > 0$

Salt concentration 2 g/l

Inflow rate = 3 l/min

Outflow rate = 3 l/min

$$\frac{ds}{dt} = (\text{rate of salt into tank}) - (\text{rate of salt out of tank})$$

$$\frac{ds}{dt} = (2 \times 3) \text{ g/min} - \frac{3s}{100} \text{ g/min}$$

$$\frac{ds}{dt} = 6 - 0.03s$$

$$\frac{ds}{dt} = \frac{600 - 3s}{100}$$

$$100 ds = (600 - 3s) dt$$

$$\int \frac{ds}{600 - 3s} = \int \frac{1}{100} dt$$

$$\int \frac{1}{600 - 3s} ds = \int \frac{1}{100} dt$$

$$\frac{s}{600} + \frac{1}{3} \ln |600 - 3s| = \frac{t}{100} + c$$

$$\ln |600 - 3s| = \frac{-3t}{100} + c$$

$$600 - 3s = e^{-3t/100} \cdot e^c$$

$$600 - 3s = A e^{-3t/100}$$

$$\frac{-3s}{3} = \frac{A e^{-3t/100} - 600}{-3}$$

$$s(t) = \frac{600 - A e^{-3t/100}}{3}$$

For pure water $s(0) = 0$

$$0 = \frac{600 - A}{3} \Rightarrow A = 600$$

$$s(t) = 200 - 200 e^{-3t/100}$$

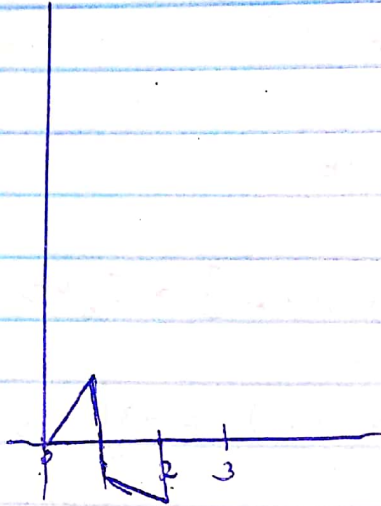
b) $s(t)$ is a decreasing function because it is a negative exponent of time.

c) ~~as~~ $s(t)$ tends to 200 the constant because as $t \rightarrow \infty$ the exponent part tends to zero.

$$\begin{aligned} d) s(t) &= 200 - 200 e^{-3t/100} = 200 - 200 e^{-0.03t} \\ &= 200 - 200 (e^{+0.03} \cdot e^{-t}) \\ &= 200 - 200 (1.03 e^{-t}) \\ s(t) &= 200 - 206 (e^{-t}) \end{aligned}$$

$$4. a) \quad y(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 \leq t \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$2-t = -t+2$$



In terms of step functions

$$y(t) = t [u_1(t) - u_1(t)] + 2-t [u_1(t) - u_2(t)]$$

from time 0 - 1s the function t is turned on as explained or shown above and when $t=1 \rightarrow t=2$ s, function $2-t$ is turned on.

$$y(t) = t u(t-0) + (2-t) [u(t-1) - u(t-2)]$$

$$y(t) = t(u(t) - u(t-1)) + (2-t) [u(t-1) - u(t-2)]$$

$$5. b) \quad (t + e^{2t})^2$$

$$L(t + e^{2t})^2$$

$$\text{but } (t + e^{2t})^2 = (t + e^{2t})(t + e^{2t}) \\ = t^2 + 2te^{2t} + e^{4t}$$

$$L[t^2 + 2te^{2t} + e^{4t}] = L[t^2] + L[2te^{2t}] + L[e^{4t}]$$

$$= \frac{2}{s^3} + 2 \cdot \frac{1}{(s-2)^2} + \frac{1}{s-4}$$

$$L[t^n] = \frac{n!}{s^{n+1}} + L[t^n e^{at}] = \frac{n!}{(s-a)^{n+1}} + L[e^{at}] = \frac{1}{s-a} = \frac{2}{s^3} + \frac{2}{(s-2)^2} + \frac{1}{s-4}$$

$$5. (a) \mathcal{L}^{-1} \frac{2s+4}{s^2-4s+12}$$

$$\frac{2s+4}{s^2-4s+12}$$

Roots of denominator

$$r_{1,2} = \frac{4 \pm \sqrt{4^2 - 4 \cdot 12}}{2}$$

$$= 2 \pm \frac{i6}{2}$$

$$= 2 \pm 3i$$

Simplify the denom by completing the square

$$s^2 - 4s + 12$$

$$s^2 - \left(\frac{4}{2}\right)s + 12 - \left(\frac{4}{2}\right)^2$$

$$s^2 - 4s + 8 + 4$$

$$(s^2 - 4s + 4) + 8$$

$$(s-2)^2 + 8$$

$$\frac{2(s+2)}{(s-2)^2+8} = 2 \frac{(s+2)}{(s-2)^2+8} = 2 \frac{(s+2)}{(s-2)^2+\sqrt{8}}$$

$$\mathcal{L}^{-1} \left[\frac{2(s+2)}{(s-2)^2+\sqrt{8}} \right] = \sqrt{2} e^{2t} (2 \sin 2\sqrt{2}t) + \sqrt{2} \cos(2\sqrt{2}t)$$

$$6. (b) \frac{s}{(s-4)(s^2+1)}$$

According to convolution theorem

$$(f * g)_t = \int_0^t f(t-v)g(v) dv$$

$\frac{s}{(s-4)(s^2+1)}$ is a product of 2 the Laplace transform

$$\frac{s}{s-4} \cdot \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1} \frac{1}{s^2+1} = \sin t$$

$$\mathcal{L}^{-1} \left[\frac{s}{s-4} \cdot \frac{1}{s^2+1} \right] = \mathcal{L}^{-1} \left[\frac{s}{s-4} \right] \cdot \mathcal{L}^{-1} \left[\frac{1}{s^2+1} \right] = \delta t + 4e^{4t} \cdot \sin t$$

for $t > 0$ $\delta t = 0$

therefore $4e^{4t} \sin t$

6. $E = 100 \cos 10t$
 $L = 1 \text{ H}$
 $R = 40 \Omega$
 $C = 16 \times 10^{-4} \text{ F}$
 $\phi(0) = 0$
 $i(0) = 0$

This is a series connection

$$\alpha = \frac{R}{2L} = \frac{40}{2 \times 1} = 20$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{16 \times 10^{-4}}} = 25$$

$\alpha < \omega_0$, therefore the system is underdamped

General soln

$$i(t) = e^{-\alpha t} [B_1 \cos \omega_d t + B_2 \sin \omega_d t]$$

$$= e^{-20t} [B_1 \cos 15t + B_2 \sin 15t]$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{25^2 - 20^2} = 15$$

$$i(0) = 0 = B_1$$

$$\therefore i(t) = e^{-20t} B_2 \sin 15t \quad \text{— current at any instant } t$$

$$\frac{di}{dt} = e^{-20t} \cdot 15 B_2 \cos 15t \quad i'(0) = 0 = e^{-20t}$$

We know $Q = CV$ $V = 100 \cos(10t)$ Finally because of the

$Q =$ parallel connection

$$Q = 16 \times 10^{-4} \text{ F} \cdot 100 \cos 10t$$

$$Q(t) = 0.16 \cos 10t$$

$$b. a) E = 100 \cos 10t$$

$$L = 1 \text{ H}$$

$$R = 40 \Omega$$

$$C = 16 \times 10^{-4} \text{ F}$$

$$v(0) = 0$$

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b)

Generic soln

$$i(t) = e^{-\alpha t} [B_1 \cos \omega_d t + B_2 \sin \omega_d t]$$

$$= e^{-20t} [B_1 \cos 15t + B_2 \sin 15t]$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{25^2 - 20^2} = 15$$

$$i(0) = 0 = B_1$$

$$\therefore i(t) = e^{-20t} B_2 \sin 15t \quad \text{— current at any instant } t$$

$$\frac{dq}{dt} = e^{-20t} \cdot 15 B_2 \cos 15t \quad i(0) = 0 = e^{-20 \cdot 0}$$

We know $Q = CV$ $V = 100 \cos(10t)$ Firstly because of the

~~C~~ = parallel connection

$$Q = 16 \times 10^{-4} \text{ F} \cdot 100 \cos 10t$$

$$Q(t) = 0.16 \cos 10t \quad \text{(charge at any instant)}$$